Symmetries and modes of wave fields in inhomogeneous media

2000 J. Phys. A: Math. Gen. 33 4779

(http://iopscience.iop.org/0305-4470/33/26/305)

J. Phys. A: Math. Gen. **33** (2000) 4779–4795. Printed in the UK PII: S0305-4470(00)01399-8

Symmetries and modes of wave fields in inhomogeneous media

Maciej Janowicz
Instytut Fizyki Polskiej Akademii Nauk, Al Lotników 32/46, 02-668 Warszawa, Poland
E-mail: mjanow@ifpan.edu.pl

Received 28 January 1999, in final form 13 April 2000

Abstract. In this paper the 2D Helmholtz equation with space-dependent dielectric permittivity and/or magnetic permeability is considered. The structure of the determining equations for the symmetry generator is analysed. It allows one to find many nontrivial spatial profiles of dielectric functions for which one can still separate variables in both the Helmholtz equation and corresponding Hamilton–Jacobi equation. For the case of the Dirichlet problem of the half-plane in an inhomogeneous medium the diffractive modes of the field are characterized. Several specific examples are provided.

1. Introduction

Recently, the problems connected with the propagation of radiation in artificial dielectrics have become very attractive due to the progress in photonic crystal design [1–6]. The theoretical questions involved contain, in particular, finding the appropriate modes of the electromagnetic field and solving problems of scattering of electromagnetic waves in such crystals. The research in this field deals with both 3D and 2D structures and uses both numerical and analytical methods.

In this work we study the 2D Helmholtz equation with spatially varying wavenumber from the point of view of the symmetry analysis of differential equations. This equation is known to serve as a laboratory for studying methods of mathematical physics, both exact—the foremost of which is the Wiener–Hopf technique [7,8] for diffraction problems,—and asymptotic [9,10]. Our approach consists of the application of the symmetry analysis of differential equations in order to perform the separation of variables. Using symmetry generators we show how to find characteristics of inhomogeneity of the medium (that is, the dielectric function) in which the waves propagate such that they are compatible with the boundary conditions. Therefore, the problem of finding appropriate diffractive modes of the wave field can be solved. As an important example, we shall consider the Dirichlet problem on the half-plane.

The rest of the paper is organized as follows. In section 2, the symmetry of the 2D Helmholtz equation is investigated. The connection between the dielectric function and the generators of the algebra of symmetry admitted by this equation is shown. The functions which determine the generator of symmetry are found to be the real and imaginary parts of an analytical function. Then, in section 3, we associate several nontrivial dielectric functions with corresponding symmetries and perform the separation of variables in the Helmholtz equation. In section 4 we investigate the symmetry of this equation when the dielectric constant depends on one Cartesian or polar coordinate. In section 5 the Dirichlet problem for the half-plane in an inhomogeneous medium is considered. Section 6 contains final remarks and conjectures.

0305-4470/00/264779+17$30.00

2. Symmetry of variable coefficients in Helmholtz equations

In this paper we apply the methods of point symmetries of differential equations to the Helmholtz equations. The concept of symmetry belongs to most of the important concepts of both physics and mathematics. In the case of differential equations, the methods associated with symmetry are important due to the following reasons [11–15]:

(a) if one finds all the symmetries of a system of partial differential equations one can obtain new solutions from other solutions found earlier—sometimes it is possible to obtain important solutions from a very trivial one (for instance, one can obtain the fundamental solution (propagator) of the free time-dependent Schrödinger equation from the constant solution);
(b) one can classify the solutions with respect to the symmetry;
(c) in the case of ordinary differential equations, one can effectively decrease the order of the equation, and sometimes reduce the problem to quadratures (indeed, all known methods to solve ordinary differential equations are implicit or explicit applications of symmetries);
(d) in the case of linear partial differential equations the symmetries are useful in revealing and classifying the coordinate systems in which the separation of variables is possible;
(e) in the case of nonlinear partial differential equations the application of the symmetries often leads to finding special solutions via the so-called symmetry (similarity) reduction;
(f) knowledge of symmetries enables one to find the conservation laws for Lagrangian systems via Noether's theorem.

In this paper we shall restrict ourselves to the so-called point symmetries of the Helmholtz equations. The basic definitions and procedures are described very briefly in the following paragraphs.

Let x^n denote a collection of independent variables and let u^α be a set of dependent variables. Let us consider the one-parameter transformation to new variables $x^{\star n} = x^{\star n}(x^i, u^\beta; \epsilon), u^{\star\alpha}(x^i, u^\beta; \epsilon)$, where ϵ is a parameter of the transformation. The corresponding infinitesimal transformation

$$x^{\star n} = x^n + \epsilon \xi^n(x^i, u^\beta) + \cdots \tag{1}$$

and

$$u^{\star\alpha} = x^{\star\alpha} + \epsilon \chi^\alpha(x^i, u^\beta) + \cdots \tag{2}$$

are generated by the operator

$$X = \sum_n \xi^n(x^i, u^\beta) \frac{\partial}{\partial x^n} + \sum_\alpha \chi^\alpha(x^i, u^\beta) \frac{\partial}{\partial u^\alpha}. \tag{3}$$

We are not able to say anything about the symmetry of a differential equation if we do not know how the partial derivatives of u^α with respect to x^n transform under the action of the transformation given by (1) and (2). That is, we have to compute the *prolongation* or *extension* of the transformation to derivatives of dependent variables. At this point, it is very convenient to introduce the operator of the total derivative:

$$\frac{\mathrm{D}}{\mathrm{D}x^n} = \frac{\partial}{\partial x^n} + u^\alpha_{,n} \frac{\partial}{\partial u^\alpha} + u^\alpha_{,nm} \frac{\partial}{\partial u^\alpha_{,m}} + \cdots \tag{4}$$

where $u^\alpha_{,n} = \partial u^\alpha / \partial x^n$ etc. With the help of the total derivative we can write the differentials of independent and dependent variables in a simple form:

$$\begin{aligned} \mathrm{d}u^{\star\alpha} &= \left(\frac{\partial u^\alpha}{\partial x^i} + \epsilon \frac{\mathrm{D}\chi^\alpha}{\mathrm{D}x^i} \right) \mathrm{d}x^i + \cdots \\ \mathrm{d}x^{\star n} &= \left(\delta^n_m + \epsilon \frac{\mathrm{D}\xi^n}{\mathrm{D}x^m} \right) \mathrm{d}x^m + \cdots \end{aligned} \tag{5}$$

from which it follows that

$$\frac{\partial u^{\star\alpha}}{\partial x^{\star n}} = \frac{u^\alpha_{,i} + \epsilon \frac{D\chi^\alpha}{Dx^i} + \cdots}{\delta^n_m + \epsilon \frac{D\xi^n}{Dx^m} + \cdots} \delta^i_m = u^\alpha_{,n} + \epsilon \left(\frac{D\chi^\alpha}{Dx^n} - u^\alpha_{,i} \frac{D\xi^i}{Dx^n} \right) + \cdots . \qquad (6)$$

On the other hand, for the infinitesimal transformation of first derivatives we should have a formula similar to (2), that is

$$u^{\star\alpha}_{,n} = u^\alpha_{,n} + \epsilon \chi^\alpha_n + \cdots \qquad (7)$$

which means that

$$\chi^\alpha_n = \frac{D\chi^\alpha}{Dx^n} - u^\alpha_{,i} \frac{D\xi^i}{Dx^n} = \frac{D}{Dx^n}(\chi^\alpha - u^\alpha_{,i}\xi^i) + \xi^i u^\alpha_{,in}. \qquad (8)$$

The infinitesimal transformations (1)–(7) are generated by the *extended* or *prolonged* infinitesimal operator (generator) $X^{(1)}$:

$$X^{(1)} = \sum_n \xi^n(x^i, u^\beta) \frac{\partial}{\partial x^n} + \chi^\alpha(x^i, u^\beta) \frac{\partial}{\partial u^\alpha} + \sum_{n,\alpha} \chi^\alpha_n(x^i, u^\beta, u^\beta_{,n}) \frac{\partial}{\partial u^\alpha_{,n}} \qquad (9)$$

where χ^α_n is given by (8). The superscript (1) on X in (9) means that it is the infinitesimal generator prolonged to first derivatives. In a very similar way we define the second prolongation (extension) of the generator:

$$X^{(2)} = X^{(1)} + \sum_{\alpha,n,m} \chi^\alpha_{nm} \frac{\partial}{\partial u^\alpha_{,nm}} \qquad (10)$$

where the functions χ^α_{nm} can be obtained from the differential $du^{\star\alpha}_{,n}$:

$$du^{\star\alpha}_{,n} = d(u^\alpha_{,n} + \epsilon\eta^\alpha_n + \cdots) = \sum_m \left(u^\alpha_{,nm} + \epsilon \frac{D\chi^\alpha_n}{Dx^m} + \cdots \right) dx^m.$$

We obtain

$$\chi^\alpha_{nm} = \frac{D\chi^\alpha_n}{Dx^m} - \sum_i u^\alpha_{,ni} \frac{D\xi^i}{Dx^m} \qquad (11)$$

or

$$\chi^\alpha_{nm} = \frac{D}{Dx^m}\left(\chi^\alpha_n - \sum_i u^\alpha_{,ni}\xi^i \right) + \sum_i \xi^i u^\alpha_{,inm}. \qquad (12)$$

Higher-order prolongations are calculated via the transformed higher-order differentials. We are now ready to define the symmetry of a system of differential equations. Let

$$\Delta(y^a) = 0 \qquad y^a = \{x^i, u^\alpha, u^\alpha_{,i}, u^\alpha_{,ik}\} \qquad (13)$$

be such a system. A point (or Lie) symmetry of the system is such a transformation

$$y^{\star a} = y^{\star a}(y^b; \epsilon) \qquad (14)$$

which is an extension of the derivatives of a point transformation (1), (2), and which maps solutions of (13) to (other) solutions. That is, if y^a is a solution of $\Delta = 0$ then $\Delta(y^{\star a}) = 0$ also holds for all values of ϵ. It has been proved in, e.g., [11, 12] or [13], that the necessary and sufficient condition for a transformation to be a symmetry is the equality

$$X^{(N)}\Delta|_{\Delta=0} = 0 \qquad (15)$$

provided that the so-called maximal rank condition is fulfilled by Δ. This condition is equivalent to the statement that not all derivatives of Δ can vanish on $\Delta = 0$. In the above equation N denotes the order of the highest-order equation contained in the system Δ. Let us stress that the expression $X^{(N)}(\Delta)$ has to vanish only in the subspace of solutions $\Delta = 0$ and not everywhere. Equation (15) provides the effective method to find the point symmetries. They are called 'point' (or 'Lie') symmetries since the functions ξ and χ in (3) depend on the independent and dependent variables but *not* on the derivatives of u (X is thus 'local'). If X (itself and not only its prolongations) is allowed to contain the derivatives of u with respect to x, we say that Δ possesses *higher* or *generalized* or *Lie–Baecklund* symmetries [12, 13]. These higher symmetries will not be used in this paper, however.

After computing the Nth prolongation of X and writing (15) in a form specific to the given system, we obtain a linear overdetermined system of differential equations for various derivatives of ξ and χ contained in $X^{(N)}$. These equations are called *determining equations*. Usually, writing down and solving them is not very difficult but extremely laborious and tedious. In connection with this, several computer programs have been developed which enable us to write down, simplify, and sometimes solve the determining equations. It is important to note, however, that for the case of linear systems there exists a much simpler, though equivalent, method for determining symmetry. Indeed, let L be the linear differential operator such that

$$L(u(x, y)) = 0 \tag{16}$$

is the equation the symmetries of which we are looking for. Let X be the generator of a symmetry; it is again a partial differential operator. Since it has to transform solutions of (16) into (other) solutions, it must be true that

$$L(X(u(x, y))) = 0. \tag{17}$$

This means that the operators L and X must commute on the space of solutions to equation (16). This statement can also be expressed as a commutator equality:

$$[L, X](f(x, y)) = r(x, y)L(f(x, y)) \tag{18}$$

where $f(x, y)$ is an arbitrary function differentiable sufficiently many times. While the general Lie method outlined above only allows one to seek the first-order symmetry operators—which enjoy the property of being members of a Lie algebra—solving the commutator equation (18) allows one to find the higher-order (i.e., expressed by operators of the order higher than one) symmetries as well. These higher-order operators do not in general, however, form a Lie algebra. In practice, one usually restricts oneself to the case of first- and second-order symmetries. If X is a first-order symmetry, then $r(x, y)$ is simply a function of x and y. If X is of second order, then $r(x, y)$ is a first-order operator. In both cases $r(x, y)$ is established in the process of finding X. If X is a first-order symmetry generator, it can be written as

$$X = \sum_\alpha \left(\xi_\alpha \frac{\partial}{\partial x_\alpha} + \psi_\alpha \right). \tag{19}$$

It acts in the x_α space. The corresponding generator obtained with the help of the Lie method is given by

$$\sum_\alpha \left(\xi_\alpha \frac{\partial}{\partial x_\alpha} - \psi_\alpha u \frac{\partial}{\partial u} \right) \tag{20}$$

and it acts in (x_α, u) space.

Let us consider the following Helmholtz equation describing a stationary wave motion in two spatial dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k_0^2 \sigma(x, y) u = 0 \tag{21}$$

where u is a wavefunction (it can be, for instance, the z component of the electric field of a transverse-magnetic wave propagating in two dimensions), $k_0 = a_0\omega/c$ with

ω being the frequency and c the phase velocity of waves, while $\sigma(x, y)$ is a function describing macroscopically the properties of the medium in which the waves propagate; in the electromagnetic case $\sigma(x, y) = \epsilon(x, y) \cdot \mu(x, y)$. In what follows, σ will be called the dieletric function since usually we have to deal with nonmagnetic natural or artificial materials. It is assumed that the natural wavelength $2\pi c/\omega_0$ is much larger than both the dimensions of, and distances between, the scatterers which the artificial dielectric is made of. The parameter a_0 is of the dimension of length and has been introduced to make the variables x, y, r dimensionless.

Let us investigate the symmetries of the Helmholtz equation (for constant σ they have been obtained in [14]). There are two independent variables and one dependent variable. We have $x^1 = x$, $x^2 = y$, $\xi^1 = \xi$, $\xi^2 = \eta$, $u^1 = u$, $\chi^1 = \chi$. The variables ξ and η characterize explicitly how the coordinates change if a particular reference frame is changed from one to another. The function χ shows the corresponding change of the physical field. The infinitesimal symmetry generator X has the form:

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \chi(x, y, u)\frac{\partial}{\partial u}. \tag{22}$$

Its prolongations have been obtained from equations (8) and (12). Employing, e.g., the *Mathematica* package lie.m we get a system for determining equations for ξ, η, χ. From this system it follows that ξ and η do not depend on u, and that χ is equal to $\chi_0(x, y) + au$, where $\chi_0(x, y)$ is an arbitrary solution to the Helmholtz equation (the trivial symmetry connected with χ_0 expresses the superposition principle; it will not be of any use here), and a is an arbitrary constant. Besides, the functions ξ and η have to fulfil the following equations:

$$\xi_x(x, y) = \eta_y(x, y) \tag{23}$$

$$\xi_y(x, y) = -\eta_x(x, y) \tag{24}$$

$$\xi\sigma_x + \eta\sigma_y = -2\xi_x\sigma = -2\eta_y\sigma \tag{25}$$

where the subscripts denote partial derivatives, e.g., $\eta_y = \frac{\partial\eta}{\partial y}$. The same results follow—still simpler—from equation (18) (except for the trivial symmetry expressing the superposition principle).

It is thus clear that ξ and η satisfy the Cauchy–Riemann equations which means that they are the real and imaginary parts of an analytical function. Let us also note that it is ξ and η which play a key role in both the mathematics and physics of the wave propagation since $\chi = \chi_0 + au$ has the same form for all possible refractive indices. For given σ, equations (23)–(25) give an overdetermined system from which ξ and η can be calculated. It is possible and interesting, however, to proceed in an opposite direction. That is, one may first choose an analytical function, extract its real and imaginary parts, prescribe in this way the symmetry of the problem, and find the corresponding family of allowed σ. A similar procedure can be encountered in geometrical (Hamiltonian) optics. For instance, in the design of optical instruments one is often interested in getting some peculiar properties of images of canonical objects. As an example, one can look for such a medium (i.e. such a refractive index) that the geometro-optical image of a ball is also a ball (located in the image space) [16, 17]; or, one can ask what the dielectric constant of the medium must be in order to be assured that all the rays crossing a given point (x_0, y_0) will also cross themselves at an conjugate point (x_1, y_1). These and related problems have been solved, e.g., in a book by Luneburg [16]. Their solutions consist of the characterization of $\sigma(x, y)$ by the specification of certain symmetry of solutions in terms of geometrical optics.

Here we deal with (2D) wave optics, and the symmetry of the system is most easily described in terms of the infinitesimal operators containing the information as to what kind of transformation, compared with other reference frames, leaves the system invariant.

On the other hand, it is also important to know the symmetry of the Helmholtz equation corresponding to a given family of refractive indices. In section 4 we discuss two examples which include Luneburg's solutions mentioned above and show their meaning from the point of view of symmetry.

The possibility of solving equation (25) for σ depends on our ability to solve the following system of ordinary differential equations related to (25):

$$\frac{\mathrm{d}x}{\xi(x,y)} = \frac{\mathrm{d}y}{\eta(x,y)} = -\frac{\mathrm{d}\sigma(x,y)}{R\sigma(x,y)} \tag{26}$$

where $R = 2\xi_x = 2\eta_y = \xi_x + \eta_y$. Below we give several simple examples of analytical functions for which it is easy to solve (26). After finding σ, we may also try to find the invariant form $\Theta(x, y)$ of the infinitesimal generator X [13] which is defined by

$$X(u - \Theta(x,y))|_{u=\Theta(x,y)} = 0. \tag{27}$$

The invariant form Θ is an eigenfunction of the operator $\frac{\partial}{\partial t}$ which is obtained by 'straightening' the operator $X_1 = X - \chi\frac{\partial}{\partial u}$. The straightening can be achieved by solving the system [11]:

$$X_1 s(x,y) = 0 \tag{28}$$

$$X_1 t(x,y) = 1. \tag{29}$$

The variables $s = s(x, y)$ and $t = t(x, y)$ obtained by solving equations (28) and (29) are simply new—'good', or 'symmetry-adapted'—coordinates; the Helmholtz operator written in terms of them allows the separation of variables. Before we consider several examples, let us mention that in many cases it is more convenient to start from the polar coordinates (r, ϕ) instead of (x, y). Then the Helmholtz equation reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + k_0^2\sigma(r,\phi)u = 0. \tag{30}$$

If we write the infinitesimal generator X' as

$$X' = r\zeta_1\frac{\partial}{\partial r} + \zeta_2\frac{\partial}{\partial \phi} + (\chi_0 + au)\frac{\partial}{\partial u} \tag{31}$$

where χ_0 is an arbitrary solution of the Helmholtz equation (30), we find that ζ_1 and ζ_2 again satisfy the Cauchy–Riemann equations:

$$\zeta_{1r} = \frac{1}{r}\zeta_{2\phi} \tag{32}$$

$$\zeta_{2r} = -\frac{1}{r}\zeta_{1\phi} \tag{33}$$

so that they are the real and imaginary parts of an analytical function. In addition, these functions together with $\sigma(r, \phi)$ have to satisfy the equation

$$r\zeta_1\sigma_r + \zeta_2\sigma_\phi = -2\sigma\left(\zeta_1 + \frac{\partial\zeta_2}{\partial\phi}\right). \tag{34}$$

The following remark should be noted here. Usually, if we allow the coefficients of a linear partial differential equation to vary, this results in a spoiling of the symmetry, except in some very particular cases. However, for the Helmholtz equation the family of these 'particular cases' is very large and we may associate fairly unexpected and counter-intuitive symmetry generators with an amazingly large collection of σ. This is in contrast to the case of, e.g., the time-dependent Schrödinger equation. In one spatial dimension, the time-dependent Schrödinger operator admits just four potentials for which nontrivial point symmetry exists; these are the constant, the linear, the harmonic and the inverse-square potentials. In two spatial dimensions the number of point-symmetry-admitting potentials reduces to three. Now, the number of dielectric constants for which the Helmholtz operator admits a nontrivial point symmetry is uncountable. Also, we shall see that with a given one-parameter symmetry generator which satisfies the conditions (23)–(25) or (32)–(34) one may associate many different dielectric functions such that the separation of the variables is possible.

3. Examples of the symmetries and related dielectric function

Let us start with the simplest possible case: our analytical function $w(z) = \xi(x, y) + \mathrm{i}\eta(x, y)$, where $z = x + \mathrm{i}y$, is linear, i.e.,

$$w = z_0 + b\mathrm{i}z \tag{35}$$

where z_0 is an arbitrary complex constant and b is an arbitrary real parameter. Then

$$\xi = \xi_0 - ay \qquad \eta = \eta_0 + ax. \tag{36}$$

This defines a three-parameter (ξ_0, η_0, a) symmetry algebra of the Helmholtz equation generating the 2D Euclidean group $E(2)$. We immediately realize that the corresponding σ must be a constant. This example has been elaborated in great detail in a fundamental book by Miller [14]. Below in this section we shall concentrate mainly on the one-parameter transformations.

As our first example, let us consider another simple holomorphic function $w(z) = z$. Then $\zeta_1 = r\cos\phi$, $\zeta_2 = r\sin\phi$. Let us first find the 'good' variables s and t.

Solving first the system

$$\frac{\mathrm{d}r}{r^2\cos\phi} = \frac{\mathrm{d}\phi}{r\sin\phi} \tag{37}$$

we find $s = \Psi(\sin\phi/r)$ where Ψ is an arbitrary differentiable function of its argument. The simplest choice of Ψ is $\Psi(x) = x$, i.e., $s = \sin\phi/r$. This solution is thus not unique: equation (37) has infinitely many solutions, but they all are (arbitrary) functions of s.

Then we solve the equation

$$\frac{\mathrm{d}r}{r^2\cos\phi} = \frac{\mathrm{d}\phi}{r\sin\phi} = \mathrm{d}t \tag{38}$$

to find that t is given by an implicit relation: $\Phi(s, t+\cos\phi/r) = 0$, where Φ is again an arbitrary differentiable function of its two arguments. Again, the simplest choice is $t = -\cos\phi/r$. One can immediately realize that the lines of constant s and constant t are orthogonal in the plane. The above-mentioned non-uniqueness of solutions of equations (37), (38) allows for some flexibility in choosing appropriate (s, t).

The invariant form $\Theta(r, \phi)$ is then given by

$$\Theta(r, \phi) = \exp(\mathrm{i}lt)f(s) \tag{39}$$

where l is a so far arbitrary (separation) constant. Substituting $u = \Theta(s(r, \phi), t(r, \phi))$ into the Helmholtz equation (21) we obtain the following differential equation for $f(s)$:

$$f''(s) + (k_0^2\sigma r^4 - l^2)f(s) = 0. \tag{40}$$

It is thus clear that we achieve the separation of variables if it is possible to eliminate σr^4 in favour of s.

Let us check the admissible dielectric functions using equation (34). We solve

$$\frac{\mathrm{d}r}{r^2\cos\phi} = \frac{\mathrm{d}\phi}{r\sin\phi} = -\frac{\mathrm{d}\sigma}{4\sigma r\cos\phi} \tag{41}$$

and find the solution for σ in the implicit form

$$\Phi(s, \sigma r^4) = 0 \tag{42}$$

where Φ is an arbitrary smooth function of its two arguments. Equation (42) can be used to eliminate the term σr^4 from equation (40) if we can explicitly solve (42) to get σr^4 as a function of s.

To be more specific, let us consider three subcases. Let, for example, $\sigma = \sigma_0/r^4$. Then, from (39), (40) we obtain the following solution of the Helmholtz equation:

$$u = u_l = \exp(-\mathrm{i}l\cos\phi/r)\left[A_l \sin\left(\sqrt{k_0^2\sigma_0 - l^2 s}\right) + B_l \cos\left(\sqrt{k_0^2\sigma_0 - l^2 s}\right)\right]. \tag{43}$$

The general solution is obtained by integrating over l. This solution may serve as an alternative to the standard Bessel-function solution in terms of $1/r$.

To obtain a closer look at the way the separation of variables by symmetry works, let us also consider the Hamilton–Jacobi equation corresponding to equation (21) with $\sigma r^4 = \sigma_0\alpha(s)$, which results from solving the implicit relation (42). Thus, let us try to solve approximately the Helmholtz equation in terms of the Luneburg–Kline series:

$$u \approx \exp[\mathrm{i}k_0(\mathcal{A}^{(0)} + k_0^{-1}\mathcal{A}^{(1)} + k_0^{-2}\mathcal{A}^{(2)} + \cdots)]. \tag{44}$$

Expanding and keeping only the terms of the highest order in k_0 we obtain the Hamilton–Jacobi equation which is the basic mathematical formula for expressing the geometro-optical approximation:

$$\left(\frac{\partial\mathcal{A}^{(0)}}{\partial x}\right)^2 + \left(\frac{\partial\mathcal{A}^{(0)}}{\partial y}\right)^2 = \sigma(x, y). \tag{45}$$

Let us write simply $\mathcal{A}$ instead of $\mathcal{A}^{(0)}$. This function is usually called an eikonal. Now let $\sigma = r^{-4}\sigma_0\alpha(s)$. Using the Helmholtz equation, we find that in terms of new, symmetry-adapted variables s and t, the Hamilton–Jacobi equation for the eikonal acquires the trivial form

$$\mathcal{A}_s^2 + \mathcal{A}_t^2 = \sigma_0\alpha(s) \tag{46}$$

where $\mathcal{A}_s = \partial\mathcal{A}/\partial s$, $\mathcal{A}_t = \partial\mathcal{A}/\partial t$. The corresponding Hamiltonian is

$$H = p_s^2 + p_t^2 - \sigma_0\alpha(s) \tag{47}$$

where p_s and p_t are the momenta canonically conjugated to s and t. In the case of $\alpha(s) = 1$ we can easily solve the corresponding Hamilton equations:

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{\partial H}{\partial p_s} \qquad \frac{\mathrm{d}p_s}{\mathrm{d}\tau} = -\frac{\partial H}{\partial s} \tag{48}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial H}{\partial p_t} \qquad \frac{\mathrm{d}p_t}{\mathrm{d}\tau} = -\frac{\partial H}{\partial t} \tag{49}$$

where τ is a parameter along the ray. Upon solving these equations the variable τ may be eliminated in terms of s and t so that we find the equation of the ray trajectories:

$$(t - t_0) = \frac{p_{t0}}{p_{s0}}(s - s_0) \tag{50}$$

where s_0 and t_0 just specify where we start to draw the ray, while p_{s0} and p_{t0} depend on the 'initial' value of the eikonal $\mathcal{A} = \mathcal{A}_0$ on the subset Y of the plane from which the rays start, $p_{s0} = \frac{\partial\mathcal{A}_0}{\partial s_0}$, $p_{t0} = \frac{\partial\mathcal{A}_0}{\partial t_0}$. The coordinates (s_0, t_0) also serve to label the points in Y. On expressing t and s in terms of, say, Cartesian coordinates, we realize that the ray trajectories are either circles or straight lines depending on the subspace of initial 'positions' (s_0, t_0) and on $\mathcal{A}_0$. In the case of $p_{s0} = 0$ equation (50) becomes meaningless, but in this case the ray is just a line of constant s. Thus, the 'good', symmetry-adapted variables s, and t are proved to be useful in geometrical optics as well.

As a second subcase let us briefly consider $\sigma = \sigma_0 \sin\phi/r^5$. Then the mode functions $u = u_l$ take the form

$$u_l = \exp\left(-\mathrm{i}l\frac{\cos\phi}{r}\right)\left[A_l \mathrm{Ai}\left(\frac{l^2 - k_0^2\sigma_0 s}{(-k_0^2\sigma_0)^{2/3}}\right) + B_l \mathrm{Bi}\left(\frac{l^2 - k_0^2\sigma_0 s}{(-k_0^2\sigma_0)^{2/3}}\right)\right] \tag{51}$$

where Ai and Bi are Airy functions.

The Hamilton–Jacobi here takes the form

$$\mathcal{A}_s^2 + \mathcal{A}_t^2 = \sigma_0 s \tag{52}$$

so that the equation of a ray trajectory is quadratic:

$$s = s_0 + 2p_{s0}\frac{t - t_0}{2p_{t0}} + 2\sigma_0\left(\frac{t - t_0}{2p_{t0}}\right)^2 \tag{53}$$

only if $p_{t0} \neq 0$. If, on the other hand, $p_{t0} = 0$ then the rays are simply the curves of constant t.

The other subcase for which we can obtain the mode functions explicitly in terms of known special functions is $\sigma = \sigma_0 s^{-2} = \sigma_0 r^2/\sin\phi$. Then

$$u_l = \exp\left(-\mathrm{i}l\frac{\cos\phi}{r}\right)\sqrt{s}(A_l I_{-\mu i}(ls) + B_l I_{\mu i}(ls)) \tag{54}$$

where $I_{\pm\mu i}(ls)$ are the modified Bessel functions and $\mu = \frac{1}{2}\sqrt{4k_0^2\sigma_0 - 1}$. The Hamiltonian function for the corresponding geometro-optical problem reads

$$H = p_s^2 + p_t^2 - \sigma_0/s^2. \tag{55}$$

The Hamilton equations are again solvable (the equations for s and p_s are integrated by the 'energy' integral) and result in the equations for trajectories (rays):

$$\frac{t - t_0}{2p_{t0}} = \frac{\pm 1}{2\sqrt{\sigma_0}}\frac{\sqrt{1 + c_1 s^2}}{c_1} + c_2 \tag{56}$$

where c_1 and c_2 are arbitrary constants.

As a second example let us consider the Zhukovski function $w(z) = \frac{1}{2}(z + \frac{1}{z})$ so that $\zeta_1 = \frac{1}{2}(r + \frac{1}{r})\cos(\phi)$, $\zeta_2 = \frac{1}{2}(r - \frac{1}{r})\sin(\phi)$. Let us determine s and t. First we solve the system

$$\frac{\mathrm{d}r}{r\frac{1}{2}(r + r^{-1})\cos\phi} = \frac{\mathrm{d}\phi}{\frac{1}{2}(r - r^{-1})\sin\phi} \tag{57}$$

to obtain the solution

$$s = \Psi(r\sin\phi/(r^2 + 1)) \tag{58}$$

where Ψ is an arbitrary function. It is convenient to choose $s = r\sin\phi/(r^2 + 1) = y/(x^2 + y^2 + 1)$. Taking advantage of this s on intermediate steps of the calculation, we solve the equation for t:

$$\frac{1}{2}(r^2 + 1)\cos\phi\frac{\partial t}{\partial r} + \frac{1}{2}(r - r^{-1})\sin\phi\frac{\partial t}{\partial\phi} = 1, \tag{59}$$

the result being $t = \arctan(\frac{r^2-1}{2r\cos\phi}) = \arctan(\frac{x^2+y^2-1}{2x})$. Writing u as a product $\exp(ilt)f(s)$ we obtain the following equation for the function $f(s)$:

$$f''(s) + \frac{8s}{4s^2-1}f'(s) - \left(\frac{k_0^2\sigma(1+r^2)^2}{4s^2-1} + \frac{4l^2}{(4s^2-1)^2}\right)f(s) = 0. \tag{60}$$

Equation (60) will become an ordinary differential equation for $f(s)$ provided that σ is such that the term $\sigma(1+r^2)^2$ can be eliminated in favour of some function of s, that is, $\sigma(1+r^2)^2 = \sigma_0\alpha(s)$. Let us check this conclusion by investigation of equation (34) for σ. We have

$$\frac{1}{2}r(r+r^{-1})\cos\phi\frac{\partial\sigma}{\partial r} + \frac{1}{2}(r-r^{-1})\sin\phi\frac{\partial\sigma}{\partial\phi} = -2\sigma r\cos\phi. \tag{61}$$

The solution is

$$\Phi(s, \sigma(r^2+1)^2) = 0 \tag{62}$$

where Φ is an arbitrary function of two variables. The conclusion drawn from equation (60) is thus confirmed.

The Hamilton–Jacobi equation corresponding to the Zhukovski function is somewhat more complicated than in the previous examples. In terms of the symmetry-adapted variables s and t it reads

$$\mathcal{A}_t^2 + \tfrac{1}{4}(1-4s^2)^2\mathcal{A}_s^2 = \sigma_0(1-4s^2)\alpha(s). \tag{63}$$

Nevertheless, it again allows for the trivial separation of variables ($\mathcal{A} = \mathcal{A}_1(s) + \mathcal{A}_2(t)$) and the rays again can be found in terms of elementary functions if $\alpha = 1$. In the case of $\alpha(s) = s$ or $\alpha(s) = s^2$ we have to use the elliptic integrals when trying to solve the Hamilton equations for the s variable.

It seems that the simplest $\alpha(s)$ is at the same time the most interesting and important. Indeed, let $\alpha(s) = 1$. Then the profile of the dielectric function is rather well known:

$$\sigma(r) = \frac{\sigma_0}{(1+r^2)^2}. \tag{64}$$

This is the dielectric function of the Maxwell fish-eye medium. Let us now ask the question of how large the symmetry algebra of the Helmholtz equation for this σ is. From the determining equations (32) it follows that if ζ_1 is nonzero, then it satisfies (prime denotes differentiation over r)

$$r\zeta_1\left[\frac{\sigma_0}{(1+r^2)^2}\right]' = -2(\zeta_1 + r\zeta_{1r})\frac{\sigma_0}{(1+r^2)^2}$$

from which it follows that

$$\frac{\mathrm{d}\zeta_1}{\zeta_1} = \frac{1}{r}\frac{r^2-1}{r^2+1}\,\mathrm{d}r$$

so that

$$\zeta_1 = C(\phi)(r + 1/r).$$

Making use of the Cauchy–Riemann equations (32) and (33) we find that $C(\phi)$ has to satisfy the harmonic-oscillator equation

$$C''(\phi) + C(\phi) = 0.$$

Therefore, $C(\phi) = A\cos(\phi) + B\sin(\phi)$. For vanishing ζ_1 we have simply $\zeta_2 = \text{const}$, since the Cauchy–Riemann equations must still be fulfilled. We have thus realized that the Helmholtz equations with the Maxwell fish-eye profile of the dielectric functions allow the four-parameter symmetry Lie algebra with the generators:

$$\begin{aligned} X_1 &= (r + r^{-1}) \cos\phi \frac{\partial}{\partial r} + (r - r^{-1}) \sin\phi \frac{\partial}{\partial \phi} \\ X_2 &= (r + r^{-1}) \sin\phi \frac{\partial}{\partial r} - (r - r^{-1}) \cos\phi \frac{\partial}{\partial \phi} \\ X_3 &= \frac{\partial}{\partial \phi} \\ X_4 &= u \frac{\partial}{\partial u} \end{aligned} \tag{65}$$

where the trivial symmetry corresponding to the superposition principle has been ignored. Using the first three generators, the first two of them being not 'obvious', we obtain separation of the variables. It is to be noted that, unfortunately, the slightly more 'realistic' $\sigma(r) = 1 + \sigma_0/(1 + r^2)^2$ allows for only two generators (namely X_3 and X_4 of (65)), hence the variables can be separated only in the cylindrical system of coordinates (as far as we deal with the first-order symmetry operators).

Let us return for a moment to equation (60) with $\alpha(s) = 1$. By writing $f(s) = g(s)/\sqrt{1 - 4s^2}$ $(-\frac{1}{2} < s < \frac{1}{2})$, we obtain the formula

$$g''(s) + \frac{4(1 - l^2) + k_0^2 \sigma_0 (1 - 4s^2)}{(4s^2 - 1)^2} g(s) = 0. \tag{66}$$

The solution to (66) can be expressed in terms of the hypergeometric ${}_2F_1$ function [18].

As a next example, let us now try the exponential function $w(z) = \mathrm{e}^z$. We shall work here with the functions ξ, η defined in Cartesian coordinates. We obtain $\xi = \mathrm{e}^{ax} \cos(ay)$, $\eta = \mathrm{e}^{ax} \sin(ay)$, the invariants are $s = \mathrm{e}^{-x} \sin y$, $t = -\mathrm{e}^{-x} \cos y$, the invariant form $\Theta(x, y) = \exp(-\mathrm{i}l\mathrm{e}^{-x} \cos y) f(\mathrm{e}^{-x} \sin y)$. The function $f(s)$ has to satisfy the equation

$$f''(s) + (k_0^2 \sigma \mathrm{e}^{2x} - l^2) f(s) = 0 \tag{67}$$

and σ is given by the following solution to equation (26):

$$\Phi(s, \sigma \mathrm{e}^{2x}) = 0. \tag{68}$$

If σ is such that the relation (68) can be explicitly solved so that $\sigma \mathrm{e}^{2x} = \sigma_0 \alpha(s)$, we obtain the ordinary differential equation for $f(s)$:

$$f''(s) + (k_0^2 \sigma_0 \alpha(s) - l^2) f(s) = 0. \tag{69}$$

Again, in favourable cases it can be solved in terms of known special or even elementary functions. In particular, let us take $\alpha(s) = s^{-2}$ which means that $\sigma = \sigma_0/(\sin y)^2$. The solution can be written in terms of the Bessel functions

$$f(s) = \sqrt{s}(A_l I_{-i\nu}(ls) + B_l I_{i\nu}(ls)) \tag{70}$$

where $\nu = (\frac{1}{2})\sqrt{4k_0^2\sigma_0 - 1}$.

The fourth elementary complex function which will be examined is the logarithmic one, $w(z) = \log(z)$. Taking into account the principal branch we write $\zeta_1 = \log(r)$, $\zeta_2 = \phi$. The symmetry-adapted variables are now $s = \log(r)/\phi$, $t = \log(\log(r))$.

The dielectric function σ which corresponds to the symmetry determined by ζ_1 and ζ_2 belongs to the family specified by the equation

$$\Phi(s, \sigma(r \log(r))^2) = 0. \tag{71}$$

It is convenient now to write down the invariant form as

$$\Theta(s, t) = \exp(lt) f(s) = \log(r)^l f(s). \tag{72}$$

Substitution of Θ for u in the Helmholtz equation leads to the following expression:

$$f''(s) + 2\frac{l + s^2}{s(1+s^2)} f'(s) + \frac{k_0^2 \sigma r^2 \log(r)^2 + l(l-1)}{s^2(1+s^2)} f(s) = 0. \tag{73}$$

In the simplest (and this time not very interesting) case $\sigma r^2 \log(r)^2 = \sigma_0$ the solution can again be expressed in terms of the hypergeometric ${}_2F_1$ function.

Let us once again assume that σ is such that equation (71) can be solved explicitly, so that $\sigma r^2 \log(r)^2 = \sigma_0 \alpha(s)$. Then the Hamilton–Jacobi equation is derived in the form

$$s^2(1+s^2)\mathcal{A}_s^2 + \mathcal{A}_t^2 + 2s\mathcal{A}_s\mathcal{A}_t = k_0^2\sigma_0\alpha(s) \tag{74}$$

while the corresponding Hamiltonian is

$$H = s^2(1+s^2)p_s^2 + p_t^2 + 2sp_sp_t - k_0^2\sigma_0\alpha(s). \tag{75}$$

Now, the Hamiltonian function contains the product of p_s and p_t, which was absent in the previous examples, and the separation of variables in the Hamilton–Jacobi equation is not automatic. Nevertheless, the Hamiltonian does not depend on t, which still makes it possible to solve the Hamilton equations at least in an implicit form, since the solution for p_t is very simple.

Let us finally consider the hyperbolic function $w(z) = \cosh(z)$. Then $\xi = \cosh x \cos y$, $\eta = \sinh x \sin y$. The symmetry-adapted variables are $s = \sin y / \cosh x$, $t = \arctan[\sinh(x)/\cos y]$. The admissible σ must satisfy

$$\Phi(s, \sigma \cosh^2(x)) = 0. \tag{76}$$

The function $f(s)$ is now a solution to the equation

$$f''(s) + \frac{2s}{s^2-1} f'(s) - \left(\frac{l^2 + k_0^2(s^2-1)\cosh^2(x)\sigma}{(s^2-1)^2}\right) = 0. \tag{77}$$

It is again clear that $\sigma \cosh^2(x)$ must be equal to $\sigma_0\alpha(s)$ in accordance with (76).

The simple case of $\alpha = 1$ is again distinguished since the resulting dielectric constant is one of those which characterize the so-called perfect optical instrument [16]. In this case we can obtain the general solution to equation (77) in terms of Legendre functions:

$$f(s) = A_l P_l^\nu(s) + B_l Q_l^\nu(s) \tag{78}$$

where $\nu = \sqrt{4k_0^2\sigma_0 + 1} - 1$.

The Hamilton–Jacobi equation also takes the separable form

$$(1-s^2)^2\mathcal{A}_s^2 + \mathcal{A}_t^2 + \sigma_0(s^2-1) = 0. \tag{79}$$

As in the case of the Maxwell fish-eye profile, let us ask about the most general continuous transformation which leaves the Helmholtz equations with the squared hyperbolic secant profile of σ invariant, and which is expressible in terms of the first-order differential operators. The solution to equations (24)–(26) is straightforward and leads to the conclusion that the algebra of invariance of these equations is spanned by the following generators:

$$\begin{aligned} X_1 &= \cosh(x)\sin(y)\frac{\partial}{\partial x} - \sinh(x)\cos(y)\frac{\partial}{\partial y} \\ X_2 &= \cosh(x)\cos(y)\frac{\partial}{\partial x} + \sinh(x)\sin(y)\frac{\partial}{\partial y} \\ X_3 &= \frac{\partial}{\partial y} \\ X_4 &= u\frac{\partial}{\partial u}. \end{aligned} \tag{80}$$

The first two generators provide a tool for the nontrivial and perhaps somewhat amazing separation of variables, one of which has been provided above.

4. Symmetry properties in the case of σ depending on one Cartesian or cylindrical coordinate

In many cases σ is a function of x (y), or r (ϕ) only. Let us look for such σ that an interesting symmetry is admitted. By 'interesting' we mean a symmetry the generator of which does not reduce to the form $\frac{\partial}{\partial x_2}$ when σ depends on x_1 only; here x_1 and x_2 are members of the pairs (x, y), or (ρ, ϕ).

First, let σ depend only on x and not on y. Our goal is to find such σ, that the symmetry algebra is reached other than generated by $X_3 = \frac{\partial}{\partial y}$, $X_4 = \frac{\partial}{\partial u}$. Integrating equation (25) we obtain

$$\xi(x, y) = \frac{h(y)}{\sqrt{\sigma(x)}} \tag{81}$$

with the function $h(y)$ to be determined. Substituting the above expression into the Laplace equation which ξ must obey as the real part of a holomorphic function, we obtain

$$4\sigma^2(x)h''(y) + h(y)(3\sigma'^2(x) - 2\sigma(x)\sigma''(x)) = 0. \tag{82}$$

We can separate variables in this equation. Let ν^2 be the separation constant, where ν is an arbitrary real or imaginary parameter. Then we obtain for $h(y)$

$$h(y) = a\sin(\nu y) + b\cos(\nu y) \tag{83}$$

while $\sigma(x)$ satisfies the equation

$$2\sigma(x)\sigma''(x) - 3\sigma'^2(x) + 4\nu^2\sigma^2(x) = 0, \tag{84}$$

the solution of which is

$$\sigma(x) = -4C_1\nu^2\frac{1}{\cosh^2(\nu(x - C_2))} \tag{85}$$

where $C_1 > 0$. We realize that, for real ν, the coefficients a and b in (83) must be purely imaginary since both ξ and σ have to be real. From equations (83)–(85) we infer that for specific values of $\sigma(x)$ given by (85) we indeed obtain 'interesting' symmetry generators:

$$\begin{aligned} X_1 &= (\cosh(\nu(x - c))\cos(\nu y))\frac{\partial}{\partial x} + (\sinh(\nu(x - c))\sin(\nu y))\frac{\partial}{\partial y} \\ X_2 &= (\cosh(\nu(x - c))\sin(\nu y))\frac{\partial}{\partial x} - (\sinh(x - c)\cos(\nu y))\frac{\partial}{\partial y} \end{aligned} \tag{86}$$

where ν and c are arbitrary constants.

Let us now concentrate on the case of polar coordinates. We allow σ to be a function of r but not of ϕ. Then we realize that $\partial/\partial(r)(\log(r\zeta_1))$ is a function of r only, which means that ζ_1 is a product:

$$\zeta_1 = a(r)b(\phi) \tag{87}$$

with

$$a(r) = \frac{1}{r\sqrt{\sigma(r)}}. \tag{88}$$

The function ζ_1, being the real part of a holomorphic function, must satisfy the Laplace equation. If we take the Laplacian of (87), substitute (88), and separate variables in the resulting equation, we obtain

$$b''(\phi) + \nu^2 b(\phi) = 0 \tag{89}$$

for $b(\phi)$. On the other hand, $\sigma(r)$ has to satisfy the following differential equation:

$$(4 - 4\nu^2)\sigma^2(r) + 3r^2\sigma'^2(r) + 2r\sigma(r)(\sigma'(r) - r\sigma''(r)) = 0. \tag{90}$$

In the two equations above the parameter ν can be (arbitrary) real or imaginary. The solution to this equation may be conveniently written as

$$\sigma(r) = \frac{C_2}{r^2 \cosh^2[C_1 - \nu \log(r)]}. \tag{91}$$

The function $b(\phi)$ is given by

$$b(\phi) = A \sin(\nu\phi) + B \cos(\nu\phi). \tag{92}$$

The only constraint imposed on A and B is that the function $b(\phi)$ be real.

A very interesting special solution to equation (90) is obtained by putting $C_1 = 0$ in (91). Indeed, writing $4C_2 = \sigma_0$ we get

$$\sigma(r) = \sigma_0 \frac{r^{2\nu-2}}{(1 + r^{2\nu})^2} \tag{93}$$

with the corresponding refractive index $n(r)$, $(\sigma(r) = n^2(r))$

$$n(r) = n_0 \frac{r^{\nu-1}}{1 + r^{2\nu}}.$$

One can recognize in this family of $n(r)$ the profiles, parametrized by ν, which correspond to the so-called perfect optical instruments, that is, such instruments that for every point in the plane there exists a perfect conjugate point [16]. Thus, we again find a connection between the symmetries of the Helmholtz equation and geometrical optics.

5. Diffractive modes for the half-plane problem: inhomogeneous media

In this section we shall find the mode decomposition of the field u for the (Dirichlet) problem of diffraction on the perfectly reflecting half-plane in the medium with variable dielectric function (or variable potential for the corresponding quantum mechanical problem). Detailed calculations will be given for the case of the Maxwell fish-eye medium. To begin with, let us recall some basic definitions from [13]. The infinitesimal generator X is admitted by the boundary value Dirichlet problem if it leaves invariant the differential equation itself, the equation for the boundary manifold and the boundary condition on the boundary manifold. Our boundary manifold is very simple:

$$\phi - 2n\pi = 0 \tag{94}$$

for any integer n. The boundary condition to be satisfied is

$$u = 0 \tag{95}$$

on $\phi = 0$ and $\phi = 2\pi$. This means that for all r we must have

$$\zeta_2(r, 0) = \zeta_2(r, 2\pi) = 0. \tag{96}$$

It follows that ζ_2 admits the Fourier expansion:

$$\zeta_2 = \sum_m f_m r^{m/2} \sin(m\phi/2). \tag{97}$$

Then ζ_1 is given by integration of the Cauchy–Riemann equations:

$$\zeta_1 = \sum_m f_m r^{m/2} \cos(m\phi/2). \tag{98}$$

The condition (95) does not provide any further nontrivial constraints. While we have not succeded in integration of equation (34) with such general ζ_1 and ζ_2, we still believe it is of interest to find the solution for σ with particular but arbitrary m. Thus, we take only one (arbitrary) term from the series contained in (97) and (98). Upon integration of (34) we get the following σ in an implicit form:

$$\Phi\left(\frac{\sin(m\phi/2)}{a_m r^{m/2}}, \sigma(a_m r^{m+2})\right) = \Phi\left(s, \sigma\left(\frac{s}{\sin(m\phi)/2}\right)^2\right) = 0 \tag{99}$$

where $s = \sin(m\phi/2)/(a_m r^{m/2})$. In particular, σ may be a function of ϕ only. Thus we realize that the family of dielectric functions σ compatible with the boundary conditions is quite large. By 'compatible' we mean that the nontrivial symmetry generator admitted by the boundary-value problem can be found.

We assume now that $a_m = 1$. The second variable t necessary to separate variables in the Helmholtz equation is easily found:

$$t = -2r^{-m/2}\cos(m\phi/2)/m \tag{100}$$

and u is written as a product of $\exp(\mathrm{i}lt)f(s)$ to obtain the following simple differential equation for $f(s)$:

$$f''(s) + 4(k_0^2\sigma r^{m+2} - l^2)f(s) = 0 \tag{101}$$

which again becomes an ordinary differential equation when $\sigma \cdot r^{m+2}$ is eliminated in favour of s using (99). Of course, (101) should be solved with the boundary condition $f(0) = 0$ which, together with the requirements of the proper behaviour at infinity, may introduce constraints on the l and k_0. The last equation of this section can be obviously analysed using standard methods provided that a specific admissible dependence given by (99) of σ is substituted.

Let us now consider once again the example of the Maxwell fish-eye medium. In section 3 we have derived the algebra of Lie symmetries for the Helmholtz equation with $\sigma = \sigma_0/(1 + r^2)^2$. We have shown that one of the generators, denoted by X_1, contains the operator of the form $F(r)\sin\phi\frac{\partial}{\partial\phi}$. Therefore, the Dirichlet problem for the half-plane is indeed invariant under X_1, $X_1(\phi)|_{\phi=0} = X_1(\phi)|_{\phi=2\pi} = 0$. This might be somewhat amazing since the dielectric constant in the Maxwell fish-eye problem has radial dependence. Nevertheless, the 'hidden' symmetry of the Helmholtz equation with this dielectric constant makes it possible to solve the conducting half-plane problem.

For $\phi = 0$ and $\phi = 2\pi$ one has $s = 0$. Thus, the boundary condition requires that

$$f(s) = 0 \qquad \text{for} \quad s = 0$$

where $f(s)$ has to satisfy equation (60). It is more convenient to work, however, with the function $g(s) = f(s)/\sqrt{1 - 4s^2}$ which fulfils the simpler equation (66). The solution to this equation is a linear combination of hypergeometric functions ${}_2F_1$:

$$\begin{aligned} g(s) = 2^{1/4}k\sqrt{1-4s^2}\Big[&2^{-|m|/2}k^{-2|m|}(1-4s^2)^{-|m|} \\ &\times A\,{}_2F_1(\alpha - 4|m|, \beta - 4|m|, 1 - 2|m|, 1 - 4s^2) \\ &+ 2^{|m|/2}k^{2|m|}(1-4s^2)^{|m|}B\,{}_2F_1(\alpha + 4|m|, \beta + 4|m|, 1 + 2|m|, 1 - 4s^2)\Big] \end{aligned} \tag{102}$$

where A and B are constants and we have put $l = 2m$, $k = k_0\sqrt{\sigma_0}$, $\alpha = (\frac{1}{4})(1 - \sqrt{1 + k^2})$, $\beta = (\frac{1}{4})(1 + \sqrt{1 + k^2})$.

To proceed, let us specify the allowed values of m. The variable $\psi = 2t$ varies from $-\pi$ to π and has evidently the character of an angular variable. Thus, we should require that the function $\exp(\mathrm{i}lt) = \exp(\mathrm{i}m\psi)$ be the same if ψ approaches $-\pi$ and π. This means that we should consider only the integer values of m. We also observe that the collection of functions $\{\exp(\mathrm{i}m\psi)\}|_{m=-\infty}^{\infty}$ constitutes an orthogonal and complete set in the interval $(-\pi, \pi)$, and that, for given s, m must be an integer if we want $\exp(\mathrm{i}m\psi)$ to be single-valued as a function of ϕ when ψ is expressed back in terms of ϕ and r. We require the solution (102) to be regular at both $s = -\frac{1}{2}$ and $s = \frac{1}{2}$. Thus A must be equal to zero. The remaining term has to vanish when $s = 0$. Evaluating the hypergeometric function at $s = 0$ we obtain

$$g(0) = B\frac{\pi 2^{1/4(1+2|m|)}k^{1+2|m|}\Gamma(1+2|m|)}{\Gamma(\frac{1}{4}(3-\sqrt{1+k^2}+4|m|))\Gamma(\frac{1}{4}(3+\sqrt{1+k^2}+4|m|))}. \tag{103}$$

The only zeros of the above equation are those associated with the poles of one of the Γ functions present in the denominator. Since the argument of one of them is always positive, the only possibility is that

$$\frac{1}{4}\left(3-\sqrt{1+k^2}+4|m|\right) = -n \tag{104}$$

where n is any positive integer. Thus we have found a 'quantization' condition to be fulfilled by k (and, consequently, by k_0):

$$k = \sqrt{(3+4|m|+4n)^2-1} \tag{105}$$

which arises due to the boundary condition.

Thus, we have obtained the following representation of mode functions of the field which satisfies both the regularity conditions and the boundary condition:

$$u_{mn} = \exp(\mathrm{i}m\psi)f_{mn}(s) \tag{106}$$

where $s = r\sin\phi/(r^2+1)$, $\psi = 2\arctan[(r^2-1)/(2r\cos\phi)]$, and

$$f_{mn}(s) = 2^{|m|/2+1}k_{mn}^{2|m|+1}(1-4s^2)^{|m|}{}_2F_1(-\tfrac{1}{2}(1+2n), 1+2|m|+n, 1+2|m|, 1-4s^2) \tag{107}$$

with n being positive and m arbitrary integers. The last formula has been obtained by substitution of $k = k_{mn}$ given by equation (105) into (102). The integer number n specifies the number of nodes of the function $f_{mn}(s)$ in any of the open intervals $(-\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$.

Of course, in realistic arrangements the Maxwell fish-eye medium can fill only a small part of the space. The solutions found here might serve, nevertheless, as partial ones valid in some restricted spatial region.

6. Final remarks

In this work we have studied the Helmholtz equation with spatially dependent coefficients. Symmetries of this equation have been found. The infinitesimal generator of the symmetry has been expressed in terms of the real and imaginary parts of an analytical function. Thus, a connection between planar electrostatic problems and the wave propagation in an inhomogeneous media has been established. Several examples of the relation between the infinitesimal generator of the symmetry and a family of corresponding admissible dielectric functions have been given. The geometro-optical interpretation in terms of the Hamilton–Jacobi equation for the eikonal functions and the corresponding Hamiltonian has been been provided as part of several specific examples.

A Dirichlet boundary value problem for the half-plane in an inhomogeneous medium has been considered and representations of the dielectric function for which the problem can be solved, by using the symmetry methods, have been provided.

The problem under consideration admits extensions and/or completion in at least two directions. On one hand, one may study the symmetries and solutions of boundary-value problems of the stationary Maxwell equations for inhomogeneous media. It is also possible to analyse the corresponding time-dependent problems. It is, however, quite clear that the case of two spatial dimensions is rather distinguished. Indeed, the conformal transformations of the plane are much richer than their counterparts in three spatial dimensions. The 'analytical functions' in three dimensions constitute just a subset of the set of polynomials of second order in three variables x, y, z. Thus, while the problem of classification of the dielectric constants admitting separation of variables in, say, 3D Helmholtz equations is much simpler than the corresponding 2D problem touched in this work, the results are unlikely to be of interest. One might speculate at this point that the richness of the complex analysis (or even the very possibility of building it) in two dimensions is strictly connected with the symmetry of some partial differential equations, namely the Laplace and Helmholtz ones. It is to be noted that in 2D space–time one can also build a possibly very rich analysis—the hyperbolic complex analysis. The investigations of symmetry with the help of hyperbolic analytical functions may lead to interesting results in model (1 + 1)-dimensional systems including boundary-value problems. However, there is no serious candidate for an equivalent of the variable-coefficient Helmholtz equation in (1 + 1) space–time.

On the other hand, it would be possible and desirable (in the present author's opinion) to investigate the Helmholtz equation with spatially varying dielectric constants using the apparatus of second-order symmetry operators, as has similarly been done by Miller [14] for the case of constant σ. The number of interesting coordinate systems admitting separation of variables for given σ can be substantially enlarged this way.

Acknowledgments

This work has been supported by the KBN Grant 2 P03B 030 13. The determining equations of the infinitesimal generator have been obtained by the *Mathematica* program 'lie.m' by G Baumann.

References

[1] Yablonovitch E 1993 *J. Phys.: Condens. Matter* **5** 2443
[2] Yablonovitch E 1994 *J. Mod. Opt.* **41** 173
[3] Soukulis C M (ed) 1993 *Photonic Band Gaps and Localization (NATO ARW)* (New York: Plenum)
[4] Joannopoulos J D, Meade R D and Winn J N 1995 *Photonic Crystals* (Princeton, NJ: Princeton University Press)
[5] Pendry J B 1996 *J. Phys.: Condens. Matter* **8** 1085
[6] Pendry J B 1994 *J. Mod. Opt.* **41** 209
[7] Jones D 1964 *The Theory of Electromagnetism* (Oxford: Pergamon)
[8] Noble B 1958 *Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations* (London: Pergamon)
[9] Veinberg B R 1982 *Asymptotic Methods in Equations of Mathematical Physics* (Moscow: MGU) (in Russian)
[10] Maslov V P 1977 *Complex WKB Methods in Nonlinear Equations* (Moscow: Nauka) (in Russian)
[11] Stephani H 1989 *Differential Equations. Their Solution Using Symmetries* ed M MacCallum (Cambridge: Cambridge University Press)
[12] Olver P 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[13] Bluman G and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)
[14] Miller W Jr 1977 *Symmetry and Separation of Variables* (Reading, MA: Addison-Wesley)
[15] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[16] Luneburg R K 1993 *Mathematical Theory of Optics* (Warsaw: Wydawnictwo Naukowe PWN) (Polish transl.)
[17] Born M and Wolf E 1999 *Principles of Optics* (Cambridge: Cambridge University Press)
[18] Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* (US National Bureau of Standards)